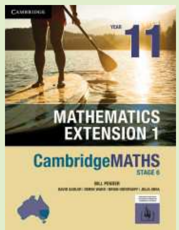
NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1
YEAR 11 COURSE

Topic summary and exercises:

With references to



(X1) **Rates of Change**

Name: ....

Initial version by H. Lam, October 2014, Revised July 2019 for latest syllabus. Last updated July 21, 2022.
Various corrections by students & members of the Mathematics Department at North Sydney Boys and Normanhurst Boys High Schools.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under CC BY 2.0.

Symbols used

Beware! Heed warning.

Ⓐ Mathematics Advanced content.

(X1) Mathematics Extension 1 content.

Literacy: note new word/phrase.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

ME11-4 applies understanding of the concept of a derivative in the solution of problems, including rates of change, exponential growth and decay and related rates of change

Syllabus subtopics

ME-C1 Rates of Change

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Mathematics Extension 1* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

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Section 1

Related rates of change

1.1 Simple numerical problems

Learning Goal(s)

Knowledge
Solve rates of change problems

Skills
Use the chain rule

Understanding
Chain rule being the key to related rates of change

By the end of this section am I able to:

10.1 Describe the rate of change of a physical quantity with respect to time as a derivative.

10.2 Find and interpret the derivative $\frac{dQ}{dt}$, given the function in the form $Q = f(t)$, for the amount of a physical quantity present at time t.

10.3 Describe the rate of change with respect to time of the displacement of a particle moving along the x-axis as a derivative $\frac{dx}{dt}$ or $\dot{x}$

10.4 Describe the rate of change with respect to time of the velocity of a particle moving along the x-axis as a derivative $\frac{d^2x}{dt^2}$ or $\ddot{x}$.

10.5 Solve problems involving related rates of change as instances of the chain rule.

10.6 Develop models of contexts where a rate of change of a function can be expressed as a rate of change of a composition of two functions, and to which the chain rule can be applied.

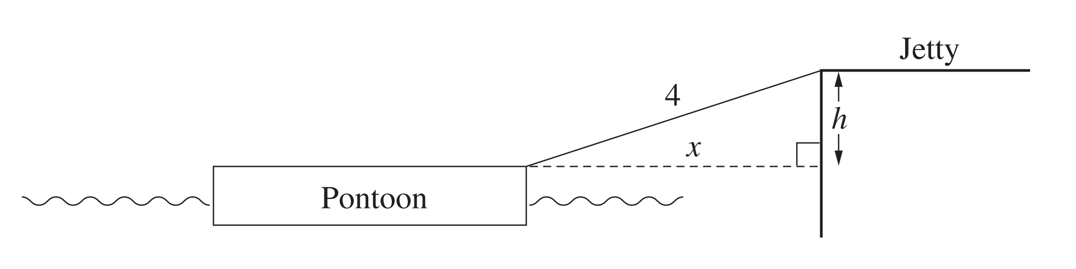
- If the relationship between quantities are related, use the .... to find the appropriate rate of change.
- There may be more than one chain involved.
- **Draw the chain diagrams required to avoid confusion.**

Important note

For questions with geometry, ... !

Example 1

[2004 Ext 1 HSC Q3] A ferry wharf consists of a floating pontoon linked to a jetty by a 4 metre long walkway. Let h metres be the difference in height between the top of the pontoon and the top of the jetty and let x metres be the horizontal distance between the pontoon and the jetty.



i. Find an expression for x in terms of h. **1**

ii. When the top of the pontoon is 1 metre lower than the top of the jetty, the tide is rising at a rate of 0.3 metres per hour. **3**

At what rate is the pontoon moving away from the jetty?

Example 2

(Pender, Sadler, Shea, & Ward, 1999, p.265) Suppose that water is flowing into a large spherical balloon at a constant rate of $50\,\text{cm}^3/\text{s}$.

(a) At what rate is the radius r increasing when the radius is 7 cm?

(b) At what rate is the radius increasing when the volume V is $4\,500\pi$ cm^3?

(c) What should the flow rate be changed to so that when the radius is 7 cm, it is increasing at 1 cm/s?

Example 3

[1993 CSSA 3U Q5] A sector of a circle with centre O and radius r cm is bounded by the radii OP and OQ, and by the arc PQ. The angle POQ is θ radians.

i. Given that r and θ vary in such a way that the area of the sector POQ has a constant value of $100\,\text{cm}^2$, show that **1**

$$\theta = \frac{200}{r^2}$$

ii. Given also that the radius is increasing at a constant rate of $0.5\,\text{cm}^{-1}$, find the rate which $\angle POQ$ is decreasing when $r = 10\,\text{cm}$. **2**

Answer: $\frac{d\theta}{dt} = -0.2$

Example 4

(Pender, Sadler, Shea, & Ward, 2000, p.263) Sand is being poured onto the top of a pile at the rate of $3\,\text{m}^3/\text{min}$. The pile always remains in the shape of a cone with semi-vertical angle 45°. Find the rate at which:

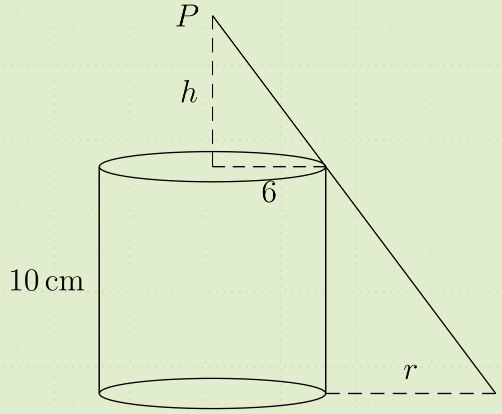
(a) the height **Answer:** $\frac{3}{4\pi}$

(b) the base area **Answer:** 3

is changing when the height is 2 metres.

Example 5

[2006 CSSA Ext 1 Trial Q5] A solid wooden cylinder of height 10 cm and radius 6 cm rests with its base on a horizontal table. A light source P is being lowered vertically downwards from a point above the centre of the top of the cylinder at a constant rate of $0.1\,\text{cms}^{-1}$.



When the light source is h cm above the top of the cylinder the shadow cast on the table extends r cm from the side of the cylinder.

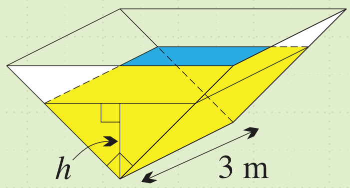
i. Show that $r = \dfrac{60}{h}$ **1**

ii. Find the rate at which r is changing when $h = 5$. **3**

Answer: $\frac{dr}{dt} = 0.24\,\text{cms}^{-1}$

Example 6

(Pender et al., 2019, Ex 16A Q15) The water trough in the diagram is in the shape of an isosceles right triangular prism, 3 metres long. A jackaroo is filling the trough with a hose at the rate of 2 litres per second.



(a) Show that the volume of water in the trough when the depth is h cm is $V = 300h^2$ cm^3

(b) Find the rate at which the water level is changing when $h = 20\,\text{cm}$, given $1\,\text{L} = 1\,000\,\text{cm}^3$.

Example 7

(Fitzpatrick, 1984, Ex 25(a)) A lamp is 6 m directly above a straight path. A man 2 m tall walks along the path away from the light at a constant speed of $1\,\text{ms}^{-1}$.

(a) At what speed is the end of his shadow moving along the path? **Answer:** $\frac{3}{2}\ \text{ms}^{-1}$

(b) At what speed is the length of his shadow increasing? **Answer:** $\frac{1}{2}\ \text{ms}^{-1}$

1.2 Two chain problems

Important note

Setting your work out neatly is crucial, especially in this section. Do not risk confusing yourself with poor setting out.

Important note

Use more than one chain where necessary.

Example 8

[2015 CSSA Ext 1 Trial] The volume of a cube is increasing at a constant rate of $100\,\text{cm}^3\text{s}^{-1}$. At what rate is the total surface area of the cube increasing when the side length of the cube is $10\,\text{cm}$?

(A) $\dfrac{5}{6}$ cm^2 per second

(B) $\dfrac{1}{3}$ cm^2 per second

(C) 40 cm^2 per second

(D) 750 cm^2 per second

Example 9

[**Legacy 2/3U Syllabus, p.72**] A spherical bubble is expanding so that its volume increases at the constant rate of $70\,\text{mm}^3/\text{s}$. What is the rate of increase of its surface area when the radius is $10\,\text{mm}$? **Answer:** $14\,\text{mm}^2\text{s}^{-1}$

1.3 Purely algebraic problems

Example 10

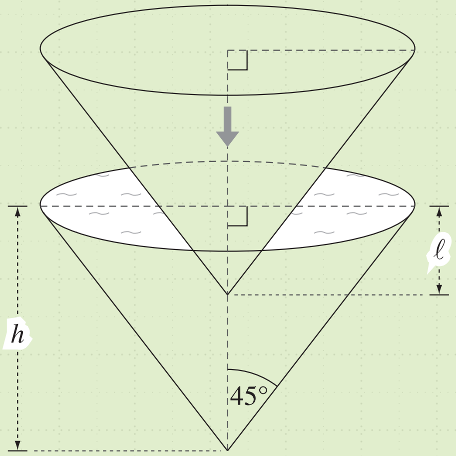
[2011 Ext 1 Q7] ⚠ The diagram shows two identical circular cones with a common vertical axis. Each cone has height h cm and semi-vertical angle $45°$.

The lower cone is completely filled with water. The upper cone is lowered vertically into the water as shown in the diagram. The rate at which it is lowered is given by

$$\frac{d\ell}{dt} = 10$$

where ℓ cm is the distance the upper cone has descended into the water after t seconds.

As the upper cone is lowered, water spills from the lower cone. The volume of water remaining in the lower cone at time t is V cm^3.



i. Show that **1**

$$V = \frac{\pi}{3}\left(h^3 - \ell^3\right)$$

ii. Find the rate at which V is changing with respect to time when $\ell = 2$. **2**

iii. Find the rate at which V is changing when the lower cone has lost $\frac{1}{8}$ of its water. Give your answer in terms of h. **2**

Further exercises (Legacy Textbooks)

Ex 7H (X1) (Pender et al., 1999)

- Q7-14

Further exercises

Ex 16A (Pender et al., 2019)

- All questions

Section 2

Rates of change proportional to current quantity

Learning Goal(s)

Knowledge
Masking time factor

Skills
Differentiating the exponential function

Understanding
Growth proportional to current quantity equals exponential growth in time

By the end of this section am I able to:

10.7 Can construct, analyse and manipulate an exponential model of the form $N(t) = Ae^{kt}$ to solve a practical growth or decay problem in various contexts (for example population growth, radioactive decay or depreciation).

2.1 Simplified model

2.1.1 Exponential growth

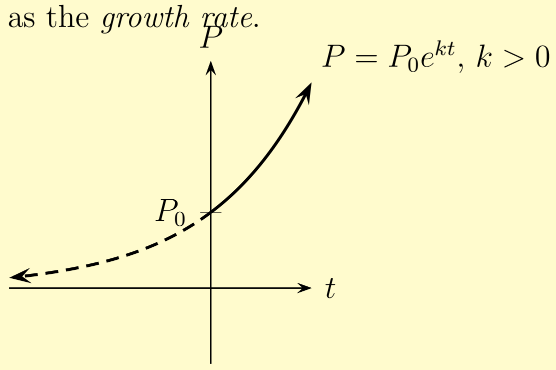
- Simplest model of population growth: growth directly proportional to current population.
- Mathematical model: $\dfrac{dP}{dt} = kP$

Theorem 1

If $\dfrac{dP}{dt} = kP$, then the solution to the differential equation is

$$P = P_0 e^{kt}$$

where P_0, $k \in \mathbb{R}^+$. k is known as the *growth rate*.



⚠ Questions will almost always require students to

1. Show that $P = P_0 e^{kt}$ is a solution to the differential equation $\dfrac{dP}{dt} = kP$.

(a) .... $P = P_0 e^{kt}$

(b) .... the time factor

2. Find the value of k to be found from the wording of the question/initial conditions.

3. Find a particular value of time and/or P given certain conditions.

Example 11

[**2004 2U HSC**] At the beginning of 1991 Australia's population was 17 million. At the beginning of 2004 the population was 20 million.

Assume that the population P is increasing exponentially and satisfies an equation of the form $P = Ae^{kt}$ where A and k are constants, and t is measured in years from the beginning of 1991.

i. Show that $P = Ae^{kt}$ satisfies $\dfrac{dP}{dt} = kP$. **1**

ii. What is the value of A? **1**

iii. Find the value of k **2**

iv. Predict the year during which Australia's population will reach 30 million. **2**

Steps

Solution

i. Differentiate P w.r.t. t and mask the time factor:

....

....

....

....

ii. Let $t = 0$:

....

....

....

....

iii. Use other conditions given:

....

....

....

....

iv. Let $P = 30$:

....

....

....

....

....

....

....

....

Example 12

Pender et al. (1999, p.480) The rabbit population P on an island was estimated to be 1 000 at the start of 1995 and 3 000 at the start of 2000.

Answer: (a) $k = \frac{1}{5}\ln 3$ (b) 5 800 rabbits (c) 10 yrs 6 mths (d) 1 760 rabbits/yr, 340 rabbits/yr

(a) Assuming natural growth, find P as a function of the time t years after the start of 1995, and sketch the graph.

(b) How many rabbits are there at the start of 2003 (answer to the nearest 10 rabbits)?

(c) When will the population be 10 000 (answer to the nearest month)?

(d) Find the rate of growth (to the nearest 10 rabbits per year):

  i. when there are 8000 rabbits,

  ii. at the start of 1997.

2.1.2 Exponential decay

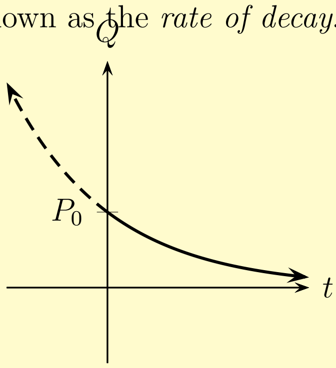
Theorem 2

If $\dfrac{dP}{dt} = -kP$, then the solution to the differential equation is

$$P = P_0 e^{-kt}$$

where $P_0 \in \mathbb{R}$, $k \in \mathbb{R}^+$. k is known as the *rate of decay*.

$P = P_0 e^{-kt}$, $k > 0$



Leave the value of k as positive to avoid arithmetic errors further down the track.

Mandatory negative sign in differential equation indicates decay.

Half life of a quantity decaying indicates the time required for the quantity to decrease to half of the original amount.

Example 13

[**2017 2U HSC Q14**] Carbon-14 is a radioactive substance that decays over time. The amount of carbon-14 present in a kangaroo bone is given by

$$C(t) = Ae^{kt}$$

where A and k are constants, and t is the number of years since the kangaroo died.

i. Show that $C(t)$ satisfies $\dfrac{dC}{dt} = kC$. **1**

ii. After 5 730 years, half of the original amount of carbon-14 is present. **2**

Show that the value of k, correct to 2 significant figures, is -0.00012.

iii. The amount of carbon-14 now present in a kangaroo bone is 90% of the original amount. **2**

Find the number of years since the kangaroo died. Give your answer correct to 2 significant figures.

Answer: 870 years

Example 14

[2013 2U HSC Q16] Trout and carp are types of fish. A lake contains a number of trout. At a certain time 10 carp are introduced into the lake and start eating the trout. As a consequence, the number of trout, N, decreases according to

$$N = 375 - e^{0.04t}$$

where t is the time in months after the carp are introduced.

The population of carp, P, increases according to

$$\frac{dP}{dt} = 0.02P$$

i. How many trout were in the lake when the carp were introduced? **1**

ii. When will the population of trout be zero? **1**

iii. Sketch the number of trout as a function of time. **1**

iv. When is the rate of increase of carp equal to the rate of decrease of trout? **3**

v. When is the number of carp equal to the number of trout? **2**

Further exercises (Legacy Textbooks)

Ex 13E (X1) (Pender et al., 1999)

- Q3-17

Ex 7G (X1) (Pender et al., 2000)

- All questions

Further exercises

Ex 16B (Pender et al., 2019)

- Q1-12

⚠ Q13 from Pender et al. (2019) is a *logistic curve* problem. This will be covered in Year 12.

2.2 Modified exponential growth and decay

Learning Goal(s)

Knowledge
Masking time factor

Skills
Differentiating the exponential function

Understanding
Growth proportional to current quantity equals exponential growth in time

By the end of this section am I able to:

10.8 Establish the modified exponential model, $\frac{dN}{dt} = k(N - P)$, for dealing with problems such as Newton's Law of Cooling or an ecosystem with a natural carrying capacity.

10.9 Solve problems involving situations that can be modelled using the exponential model or the modified exponential model and sketch graphs appropriate to such problems.

2.2.1 Modified Exponential Growth

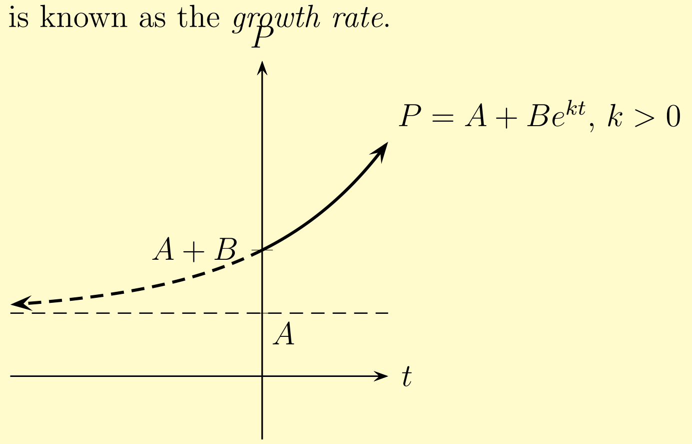
- Modified model of population growth: growth directly proportional to *difference* between the current population and a fixed constant.
- Mathematical model: $\frac{dP}{dt} = k(P - A)$

Theorem 3

If $\frac{dP}{dt} = k(P - A)$, then the solution to the differential equation is

$$P = A + Be^{kt}$$

where $A, B \in \mathbb{R}$, $k \in \mathbb{R}^+$. k is known as the *growth rate.*



Steps

⚠ Questions will almost always require students to

1. Show that $P = A + Be^{kt}$ is a solution to the differential equation $\dfrac{dP}{dt} = k(P - A)$.

   (a) .... $P = A + Be^{kt}$

   (b) .... the time factor

2. Have the value of A ready to be found by inspection.

3. Find the value of k and B to be found from the wording of the question/initial conditions.

4. Find a particular value of time and/or P given certain conditions.

2.2.2 Modified Exponential Decay (Newton's Law of Cooling)

Scenario: a boiling kettle (initially 100°C) is allowed to cool to room temperature, set at 22°C.

- Assuming the constant room temperature, the temperature T of the water will be limited to ....
- The rate at which the temperature decreases is proportional to the difference between the temperature T of the object and the temperature A of the environment
- Differential equation from wording:

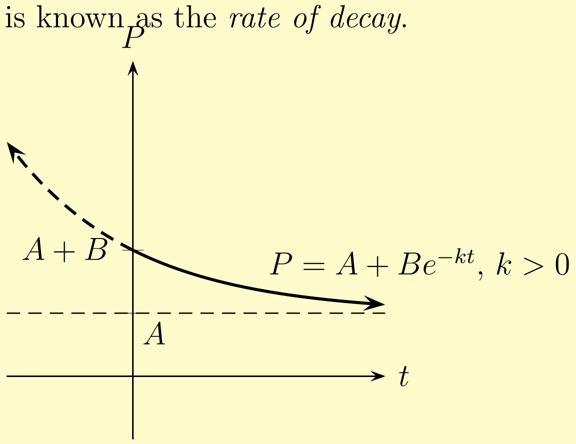
....

Theorem 4

If $\dfrac{dP}{dt} = -k(P - A)$, then the solution to the differential equation is

$$P = A + Be^{-kt}$$

where $A, B \in \mathbb{R}$, $k \in \mathbb{R}^+$. k is known as the *rate of decay*.



Example 15

(Pender et al., 2000, p.278) In a kitchen where the temperature is 20°C, Stanley takes a kettle of boiling water off the stove at time zero. Five minutes later, the temperature of the water is 70°C.

(a) Show that $T = 20 + 80e^{-kt}$ satisfies the cooling equation $\dfrac{dT}{dt} = -k(T - 20)$.

(b) Find the value of k. **Answer:** $-\frac{1}{5}\ln\frac{5}{8}$

(c) How long will it take for the water temperature to drop to 25°C?

(d) Sketch the temperature-time function.

Example 16

[**2008 Ext 1 HSC**] A turkey is taken from the refrigerator. Its temperature is 5°C when it is placed in an oven preheated to 190°C. Its temperature, T°C, after t hours in the oven satisfies the equation

$$\frac{dT}{dt} = -k(T - 190)$$

i. Show that $T = 190 - 185e^{-kt}$ satisfies both this equation and the initial condition. **2**

ii. The turkey is placed into the oven at 9 am. At 10 am the turkey reaches a temperature of 29°C. The turkey will be cooked when it reaches a temperature of 80°C. **3**

At what time to the nearest minute will it be cooked?

Example 17

[**2013 Baulkham Hills Ext 1**] When a body falls, the rate of change of its velocity v is given by $\dfrac{dv}{dt} = -k(v - 500)$, where k is a constant.

i. Show that $v = 500 - 500e^{-kt}$ is a possible solution to this equation. **1**

ii. The velocity after 5 seconds is $21\,\text{ms}^{-1}$. Find the value of k. **1**

iii. Find the velocity after 20 seconds. **1**

iv. Explain the effect on the velocity, as t becomes large. **1**

Example 18

[2013 SGS Ext 1] The city of Mongerville has a large population M. A rumour begins and t hours later the number of people who have heard the rumour is P. It is found that the rate at which the rumour spreads is proportional to the number of people who have not heard the rumour. Thus

$$\frac{dP}{dt} = k(M - P)$$

i. Show that $P = M + Be^{-kt}$ is a solution of this equation. **1**

ii. Assuming that no-one has heard the rumour initially, what is the value of B? **1**

iii. After 5 hours, half the population of Mongerville has heard the rumour. What is the value of k? **1**

iv. How long does it take for 95% of the population to have heard the rumour? Give your answer correct to the nearest minute. **1**

Answer: $k = \frac{1}{5}\ln 2$, $t = 21$ h 37 min

Further exercises (Legacy Textbooks)

Ex 7H (X1) (Pender et al., 2000)

- Q2-12

Further exercises

Ex 16C (Pender et al., 2019)

- Q1-13

NESA Reference Sheet – calculus based courses

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1},\ r \neq 1$$

$$S = \frac{a}{1 - r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

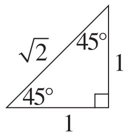
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$



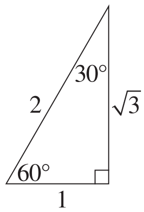
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$$\cos A = \frac{1 - t^2}{1 + t^2}$$

$$\tan A = \frac{2t}{1 - t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

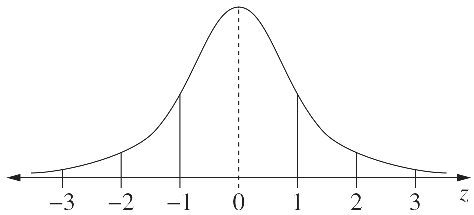
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n - r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x} p^x (1 - p)^{n - x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$z = a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta) = r^n e^{in\theta}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

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